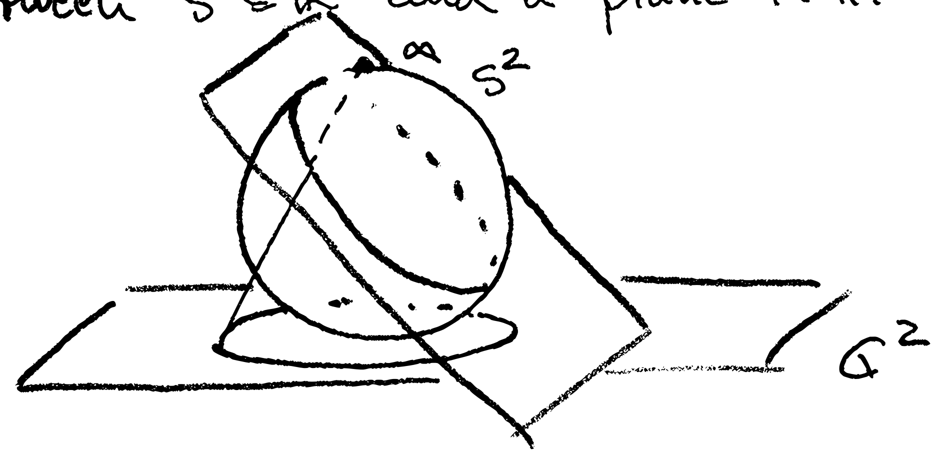
Lecture 14

- Finish discussing fixed pts + cross ratio for Möbius transformations from Lecture 13 notes.

Circles on $\mathbb{C}_\infty$.

A circle on $\mathbb{C}_\infty$ is the intersection between $S^2 \subseteq \mathbb{R}^3$ and a plane in $\mathbb{R}^3$.



A circle Γ on $\mathbb{C}_\infty$ projects to **either** a line (if $\infty \in \Gamma$) or a circle in $\mathbb{C}$.

<u>Thm 1.</u> A Möbius transf. sends circles on $\mathbb{C}_\infty$ to circles on $\mathbb{C}_\infty$.

Pf. Recall that a circle is uniquely determined by 3pts z_1, z_2, z_3 on it and the circle is a line $\Leftrightarrow$ z_1, z_2, z_3 colinear. We first show that the inverse image of $\mathbb{R}$ under any Möbius $T(z) = \frac{az+b}{cz+d}$ is a circle. $z \in \mathbb{R} \Leftrightarrow$

$$\frac{az+b}{cz+d} = \overline{\frac{az+b}{cz+d}} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \Rightarrow$$

$$(az+b)(\bar{c}\bar{z}+\bar{d}) = (cz+d)(\bar{a}\bar{z}+\bar{b}) \Rightarrow$$

$$(a\bar{c}-\bar{a}c)|z|^2 + (a\bar{d}-\bar{b}c)z + (b\bar{c}-d\bar{a})\bar{z} + (b\bar{d}-\bar{b}d) = 0$$

Two cases:

1) $a\bar{c}-\bar{a}c = 0 \Rightarrow \operatorname{Im}(\alpha z + \beta) = 0$ which is equation of a line (i.e. circle on $\mathbb{C}_\infty$).

2) $a\bar{c}-\bar{a}c \neq 0 \Rightarrow |z|^2 + \gamma z + \bar{\gamma}\bar{z} - \delta = 0$ or $|z+\gamma|^2 = \delta + |\gamma|^2$, which is eq. for circle when $\delta + |\gamma|^2 > 0$.

We could compute $r = \delta + |\gamma|^2$ but it must be positive as otherwise $T^{-1}(\mathbb{R}) = \emptyset$ if $r < 0$ or $T^{-1}(\mathbb{R}) = \{-\gamma\}$, both of which are impossible since $T: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a homeomorphism.

Now, let T be a Möbius and Γ a circle. Choose $z_1, z_2, z_3 \in \Gamma$ and let $w_1 = T(z_1)$, $w_2 = T(z_2)$, $w_3 = T(z_3)$. Let T_1, T_2 be cross ratios (z, z_1, z_2, z_3), (w, w_1, w_2, w_3)

Then $T = T_2^{-1} \circ T_1$ (by 3 f.p. Thm)

By def, $z_1, z_2, z_3 \in T_1^{-1}(\mathbb{R})$ and hence $\Gamma = T_1^{-1}(\mathbb{R})$. Moreover, $w_1, w_2, w_3 \in T_2^{-1}(\mathbb{R})$ $\Rightarrow$ the circle Γ' in $\mathbb{C}_\infty$ that contains w_1, w_2, w_3 satisfies $\Gamma' = T_2^{-1}(\mathbb{R})$ $\Rightarrow$

$$T(\Gamma) = T_2^{-1}(T_1(\Gamma)) = T_2^{-1}(\mathbb{R}) = \Gamma'.$$

$\blacksquare$

Rem. A direct consequence of the construction is that for any circles Γ, Γ' $\exists$ Möbius T s.t. $T(\Gamma) = \Gamma'$!
This T is unique once you specify $z_1, z_2, z_3 \in \Gamma$, $w_1, w_2, w_3 \in \Gamma'$ and require $T(z_j) = w_j$.

Read Symmetry and Orientation principles, but this will not be on exams.